Gaussian Distribution: Variance Scaling and Curve Width

Mathematics of Probability

Gaussian Probability Density Function (PDF)

The normal distribution PDF is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

where:

- μ is the mean (center of distribution)
- σ is the standard deviation
- σ^2 is the variance
- $(x-\mu)^2$ is the squared deviation

Scaling Role of $2\sigma^2$

The term $\frac{(x-\mu)^2}{2\sigma^2}$ in the exponent:

- Scales the squared deviation by the variance
- Controls the decay rate of the PDF
- Determines distribution width via variance

Scaling Mechanism

The division modifies sensitivity to deviations:

$$\text{Exponent} = -\underbrace{\frac{(x-\mu)^2}{2\sigma^2}}_{\text{scaled deviation}}$$

- **Large** σ^2: Smaller exponent $\rightarrow$ slower decay
- **Small** σ^2: Larger exponent $\rightarrow$ faster decay

Variance Effect on Curve Width

Case 1: High Variance ($\sigma^2 = 4$)

$$\text{At } x = \mu + 2:$$
$$\frac{(2)^2}{2 \times 4} = \frac{4}{8} = 0.5$$
$$\exp(-0.5) \approx 0.61$$

- Height at $x = \mu + 2$ is 61% of peak height
- Slow decay $\Rightarrow$ wider spread
- Probability mass distributed farther from mean

Case 2: Low Variance ($\sigma^2 = 1$)

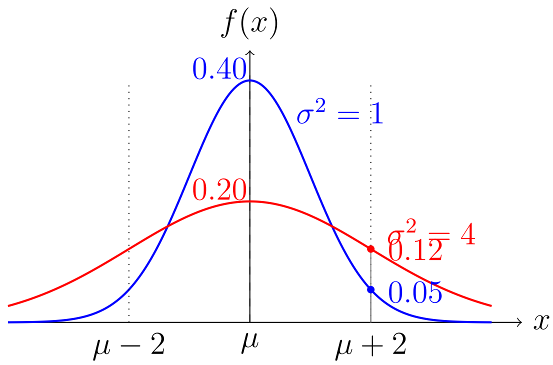
$$\text{At } x = \mu + 2:$$
$$\frac{(2)^2}{2 \times 1} = \frac{4}{2} = 2$$
$$\exp(-2) \approx 0.14$$

- Height at $x = \mu + 2$ is 14% of peak height
- Fast decay $\Rightarrow$ narrow concentration
- Probability mass focused near mean

Visual Comparison

Property	$\sigma^2 = 4$ (Wide)	$\sigma^2 = 1$ (Narrow)
Standard deviation (σ)	2	1
Peak height at $x = \mu$	$\frac{1}{2\sqrt{2\pi}} \approx 0.20$	$\frac{1}{\sqrt{2\pi}} \approx 0.40$
Height at $x = \mu + 2$	$0.20 \times 0.61 \approx 0.12$	$0.40 \times 0.14 \approx 0.05$
Relative height at $\mu + 2$	61%	14%

Graphical Representation



Why $2\sigma^2$?

The specific form ensures:

- σ corresponds to inflection points
- PDF integrates to 1 (normalization)
- Consistency with empirical rule:
  - 68% within $\mu \pm \sigma$
  - 95% within $\mu \pm 2\sigma$

Key Insight

The scaling term $\frac{(x-\mu)^2}{2\sigma^2}$ creates variance-dependent sensitivity:

$$\text{High } \sigma^2 \rightarrow \text{deviation sensitivity } \downarrow \rightarrow \text{wider curve}$$
$$\text{Low } \sigma^2 \rightarrow \text{deviation sensitivity } \uparrow \rightarrow \text{narrower curve}$$

This makes σ the natural scale for measuring deviations in normally distributed data.